Math 250 3.4 Product and Quotient Rules, Population Growth Rates

Objectives

1) Find the derivative of a product using the product rule $\frac{d}{dx}\left(f(x)\cdot g(x)\right)=f'(x)\cdot g(x)+f(x)\cdot g'(x)$

2) Simplify an expression (eventually resulting from a derivative) by factoring
   a. GCF of each numerator
   b. Least power (GCF) of each numerator
   c. LCM of each denominator

3) Find the derivative of a product of three factors
   $$\frac{d}{dx}\left(f(x)\cdot g(x)\cdot h(x)\right)=f'(x)\cdot g(x)\cdot h(x)+f(x)\cdot g'(x)\cdot h(x)+f(x)\cdot g(x)\cdot h'(x)$$

4) Find the derivative of a quotient using the quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{[g(x)]^2}$
   a. CAUTION: The order of the numerator is important – backward, and you have a sign error.
   b. CAUTION: It is illegal to cancel $g(x)$ from the first term, because there is no $g(x)$ in the second term also.

5) Where applicable, rewrite function before differentiating so the constant multiple rule applies but not the quotient rule.

6) Use population growth models to
   a. Calculate the instantaneous rate of change
   b. Approximate when the rate of change is greatest.
   c. Find the steady-state population

7) Finding derivatives using multiple rules

Examples and Practice:

1) Prove the product rule using the definition of the derivative. Explain why the derivative of the product is not the product of the derivatives.

2) Simplify by factoring: $\frac{15}{4}x^3(x+1)^{5/2}+\frac{15}{2}x^2(x+1)^{3/2}$

3) Differentiate

   a. $f(x)=\dfrac{x\sin x}{3}$

   b. $g(\theta)=5\sin\theta\cos\theta$

4) Differentiate without multiplying first (use the 3-way product rule): $f(x)=(x^6-4x^2+1)^3$

5) $f(x)=\dfrac{x^3+5x+3}{x^2-1}$

   a. Find $f'(x)$
   b. Find the equation of the tangent line to *f* at $x=2$

6) Re-write each function so that the quotient rule will not be necessary when finding its derivative.

   a. $y=\dfrac{5x^2-3}{4}$

   b. $y=\dfrac{10}{3x^5}$

7) The population of a culture of cells increases and approaches a constant level (called the steady state or carrying capacity). The population is modeled by the function $p(t)=400\left(\dfrac{t^2+1}{t^2+4}\right)$, where *t* represents time measured in hours, and *p* is the number of cells at time *t*. The rate of change of the population is called the growth rate.

   a. Use GC to graph the population function.
   b. Find the derivative of the population function.
   c. Graph the derivative of the population function, and estimate the time when the growth rate is greatest.
   d. Use the graph of the population function to estimate the steady-state population.

8) Differentiate.

   a. $f(x)=\dfrac{(3x^2-2x^{-1})(x^3+5x)}{2}$

   b. $f(x)=\dfrac{(2\sqrt{x}+1)(2-x)}{(x^2+3x)}$

   c. $f(x)=\dfrac{x\cos x}{x^3+2}$

Math 250

Goal of Product Rule: Find derivative $\frac{d}{dx}[f(x)\,g(x)]$

$\boxed{\text{* CAUTION *} \quad \text{This is NOT } f'(x)\cdot g'(x).}$

(1) Product Rule: MEMORIZE

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Proof of Product Rule: $h(x) = f(x)\cdot g(x)$

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$ definition of derivative

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}$$ definition of $h(x)$.

MAGIC: Add and Subtract a useful term (net effect 0):

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) \overbrace{- f(x+\Delta x)\cdot g(x) + f(x+\Delta x)\cdot g(x)} - f(x)\,g(x)}{\Delta x}$$

Separate into terms that form $f'(x)$ and $g'(x)$ using the properties of limits:

$$= \lim_{\Delta x \to 0} f(x+\Delta x)\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x \to 0} g(x)\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \left[\lim_{\Delta x \to 0} f(x+\Delta x)\right]\cdot\left[\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \left[\lim_{\Delta x \to 0} g(x)\right]\cdot\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$\downarrow$ $\quad$ $\downarrow$ defn $\quad$ $\downarrow$ $\quad$ $\downarrow$ defn

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

M250 3.4

The derivative of the product $f(x) \cdot g(x)$ is given by the product rule

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) g'(x)$$

but the product of the derivatives is $f'(x) \cdot g'(x)$, which is only one term and does not involve $f(x)$ or $g(x)$ as the product rule does.

* CAUTIONS *

1) You will often need many sets of parentheses in this section and the next. Be careful to write neatly, and make sure each set is a pair. (no orphans)

2) The result will often be complicated and messy. You must clean up the mess to get the correct final answer.

- No negative exponents in final answer
- Factor out the GCF of any numerators.
- Factor out the LCM of any denominators
- Factor out the least power (GCF) of any common expressions.

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② Simplify by factoring: $\frac{15}{4}x^3(x+1)^{5/2} - \frac{15}{2}x^2(x+1)^{3/2}$

$\left.\begin{array}{l} \text{GCF of numerators } 15 \,\&\, 15 : 15 \\ \text{LCM of denominators } 4 \,\&\, 2 : 4 \\ \text{GCF of variables: } x^2(x+1)^{3/2} \end{array}\right\}$ factor out $\frac{15}{4}x^2(x+1)^{3/2}$

$$= \frac{15}{4}x^2(x+1)^{3/2} \cdot \left[\frac{\frac{15}{4}x^3(x+1)^{5/2}}{\frac{15}{4}x^2(x+1)^{3/2}} - \frac{\frac{15}{2}x^2(x+1)^{3/2}}{\frac{15}{4}x^2(x+1)^{3/2}} \right]$$

Factor out means divide inside → in Math 45, we do this in our heads!

$$= \frac{15}{4}x^2(x+1)^{3/2}\left[x(x+1)^{5/2-3/2} - 2 \right]$$

$\frac{15}{2} \div \frac{15}{4} = \frac{15}{2} \cdot \frac{4}{15} = 2$

$$= \frac{15}{4}x^2(x+1)^{3/2}\left[x(x+1) - 2 \right]$$

$$= \frac{15}{4}x^2(x+1)^{3/2}\left[x^2 + x - 2 \right]$$

trinomial factors!

$$= \boxed{\frac{15}{4}x^2(x+1)^{3/2}(x+2)(x-1)}$$

Example: Bonus: version of #2, but worse...

(2) Factor $\underbrace{\frac{15}{2}x^4(x^2+1)^{5/2}}_{\text{1st term}} + \underbrace{\frac{45}{4}x^2(x^2+1)^{3/2}}_{\text{2nd term}} - \underbrace{\frac{3}{2}x(x^2+1)^{1/2}}_{\text{3rd term}}$

GCF of 15, 45, 3 is 3.
LCM of 2, 4, 2 is 4
least power $x(x^2+1)^{1/2}$

} factor out $\frac{3}{4}x(x^2+1)^{1/2}$

$$= \frac{3}{4}x(x^2+1)^{1/2}\left[\frac{\frac{15}{2}x^4(x^2+1)^{5/2}}{\frac{3}{4}x(x^2+1)^{1/2}} + \frac{\frac{45}{4}x^2(x^2+1)^{3/2}}{\frac{3}{4}x(x^2+1)^{1/2}} - \frac{\frac{3}{2}x(x^2+1)^{1/2}}{\frac{3}{4}x(x^2+1)^{1/2}}\right]$$

$$= \frac{3}{4}x(x^2+1)^{1/2}\left[\underbrace{10x^3(x^2+1)^2 + 15x(x^2+1)}_{\text{if we need to set this expression}=0} - 2\right]$$

if we need to set this expression = 0,
FOIL $(x^2+1)^2$
dist $15x$
combine

$$= \frac{3}{4}x(x^2+1)^{1/2}\left[10x^3(x^4+2x^2+1) + 15x^3 + 15x - 2\right]$$

$$= \frac{3}{4}x(x^2+1)^{1/2}\left[10x^7 + 20x^5 + 10x^3 + 15x^3 + 15x - 2\right]$$

$$= \boxed{\frac{3}{4}x(x^2+1)^{1/2}\left[10x^7 + 20x^5 + 25x^3 + 15x - 2\right]}$$

Math 250

(3) a) $f(x) = \frac{x \sin x}{3}$

notice $\frac{1}{3}$ is a constant multiple

$$f(x) = \frac{1}{3} \cdot (\underbrace{x \cdot \sin x})$$

Both are functions of x — need Product Rul

$$f'(x) = \frac{1}{3}\Big(\underbrace{\frac{d}{dx}(x)}_{\text{derivative}} \cdot \underbrace{\sin x}_{\text{no derivative}} + \underbrace{\frac{d}{dx}(\sin x)}_{\text{derivative}} \cdot \underbrace{x}_{\text{no derivative}} \Big)$$

$$f'(x) = \frac{1}{3}\left(1 \cdot \sin x + \cos x \cdot x \right)$$

$$f'(x) = \boxed{\frac{1}{3}(\sin x + x \cos x)}$$

b) $g(\theta) = 5 \underbrace{\sin\theta \cos\theta}$

Both are functions of the variable $\theta \Rightarrow$ Product Rule

Notice 5 is a constant multiple.

$$g'(\theta) = 5 \cdot \Big(\underbrace{\frac{d}{d\theta}(\sin\theta)}_{\text{derivative}} \cdot \underbrace{\cos\theta}_{\text{no derivative}} + \underbrace{\frac{d}{d\theta}(\cos\theta)}_{\text{derivative}} \cdot \underbrace{\sin\theta}_{\text{no derivative}} \Big)$$

$$g'(\theta) = 5\left(\cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta \right)$$

$$g'(\theta) = \boxed{5(\cos^2\theta - \sin^2\theta)}$$

Wide Awake? That's a trig identity!

$$g'(\theta) = \boxed{5\cos(2\theta)}$$

M250

yes If there are 3 factors; product rule extends to:

$$\frac{d}{dx}[f(x)\cdot g(x)\cdot h(x)]$$

$$= \underbrace{f'(x)}_{\frac{d}{dx}\text{ 1st only}}\cdot g(x)\cdot h(x) + f(x)\cdot \underbrace{g'(x)}_{\frac{d}{dx}\text{ 2nd only}}\cdot h(x) + f(x)\cdot g(x)\cdot \underbrace{h'(x)}_{\frac{d}{dx}\text{ 3rd only}}$$

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(4) $f(x) = (x^6 - 4x^2 + 1)^3$ Find $f'(x)$.

Note: For those who've had calculus before, we don't know the chain rule yet! We must use the product rule.

$$f(x) = \underbrace{(x^6 - 4x^2 + 1)}\underbrace{(x^6 - 4x^2 + 1)}\underbrace{(x^6 - 4x^2 + 1)}$$

3-way product rule:

$$f'(x) = \left[\frac{d}{dx}(x^6 - 4x^2+1)\right]\cdot(x^6-4x^2+1)\cdot(x^6-4x^2+1)$$

$$+ (x^6-4x^2+1)\cdot\left[\frac{d}{dx}(x^6-4x^2+1)\right]\cdot(x^6-4x^2+1)$$

$$+ (x^6-4x^2+1)\cdot(x^6-4x^2+1)\cdot\left[\frac{d}{dx}(x^6-4x^2+1)\right]$$

Useful observation: These are 3 ways of writing the same thing!

$$f'(x) = 3\cdot\left[\frac{d}{dx}(x^6-4x^2+1)\right]\cdot(x^6-4x^2+1)^2$$

$$= 3(6x^5 - 8x)(x^6-4x^2+1)^2$$

↑ factor GCF $2x$

$$\boxed{f'(x) = 6x(3x^4-4)(x^6-4x^2+1)^2}$$

Goal of Quotient Rule: Find derivative $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$

$\boxed{\text{* CAUTION *} \quad \text{This is not } \frac{f'(x)}{g'(x)}}$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

MEMORIZE

If $g(x)$ is called "low" and $f(x)$ is called "high", the following rhyme can help remember the Quotient Rule:

" Low - d High minus High - d Low
Square the bottom and away we go! "

skip

Proof of Quotient Rule: $h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$ Definition of derivative

$$= \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$ Defn of $h(x)$

Clear complex fraction by multiplying by LCD $g(x+\Delta x)\cdot g(x)$

$$= \lim_{\Delta x \to 0} \frac{g(x)\cdot f(x+\Delta x) - f(x)\cdot g(x+\Delta x)}{\Delta x\cdot g(x)\cdot g(x+\Delta x)}$$

MAGIC: Add and subtract a useful term (net effect 0):

$$= \lim_{\Delta x \to 0} \frac{g(x)\cdot f(x+\Delta x) - f(x)g(x) - f(x)g(x+\Delta x) + f(x)g(x)}{\Delta x\cdot g(x)\cdot g(x+\Delta x)}$$

Math 250

Proof of Quotient Rule, con't.

Separate into terms that form $f'(x)$ and $g'(x)$ using the properties of limits. Factor out -1.

$$= \frac{\lim\limits_{\Delta x \to 0} \left\{ g(x) \cdot \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \right\} - \lim\limits_{\Delta x \to 0} \left\{ f(x) \cdot \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \right\}}{\lim\limits_{\Delta x \to 0} \left\{ g(x) \cdot g(x+\Delta x) \right\}}$$

$$= \frac{\left[\lim\limits_{\Delta x \to 0} g(x) \right] \cdot \left[\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] - \left[\lim\limits_{\Delta x \to 0} f(x) \right] \cdot \left[\lim\limits_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]}{\left[\lim\limits_{\Delta x \to 0} g(x) \right] \cdot \left[\lim\limits_{\Delta x \to 0} g(x+\Delta x) \right]}$$

Notice $\lim\limits_{\Delta x \to 0} g(x) = g(x)$ and $\lim\limits_{\Delta x \to 0} f(x) = f(x)$. have no Δx to replace

Notice $\lim\limits_{\Delta x \to 0} g(x+\Delta x) = g(x+0) = g(x)$.

Replace definition of derivative by derivatives.

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x)}$$

$$= \frac{g(x) \cdot f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

Math 250

The Quotient Rule can be verified, using the Chain Rule (3.7, still to come).

$$h(x) = \frac{f(x)}{g(x)} = f(x) \cdot \underbrace{[g(x)]^{-1}}$$

This is a product.

This is a function composition which requires the Chain Rule

$$h'(x) = f(x) \cdot \frac{d}{dx}\left[[g(x)]^{-1} \right] + [g(x)]^{-1} \cdot \frac{d}{dx} f(x)$$ Product rule

$$= f(x) \cdot \underbrace{(-1)[g(x)]^{-2}}_{\text{derivative of outside function}} \cdot \frac{d}{dx}[g(x)] + \frac{f'(x)}{g(x)}$$

$$= -\frac{f(x) \cdot g'(x)}{[g(x)]^2} + \frac{f'(x)}{g(x)}$$

Reverse order and find common denominator

$$= \frac{f'(x)}{g(x)} \cdot \frac{g(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{[g(x)]^2}$$

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Math 250

(5) $f(x) = \dfrac{x^3+5x+3}{x^2-1}$

Method 1: Quotient Rule + power rule

$$f'(x) = \frac{(x^2-1)\cdot\frac{d}{dx}(x^3+5x+3) - (x^3+5x+3)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(3x^2+5) - (x^3+5x+3)(2x)}{[(x-1)(x+1)]^2}$$

$$= \frac{3x^4+5x^2-3x^2-5 - 2x^4-10x^2-6x}{(x-1)^2(x+1)^2}$$

$$= \frac{x^4-8x^2-6x-5}{(x-1)^2(x+1)^2}$$

$$\boxed{f'(x) = \frac{x^4-8x^2-6x-5}{(x-1)^2(x+1)^2}}$$

check: Does $(x-1)$ or $(x+1)$ divide?

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -8 & -6 & -5 \\ & & 1 & 1 & -7 & -13 \\ \hline & 1 & 1 & -7 & -13 & -18 \end{array}$$

not $x-1$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -8 & -6 & -5 \\ & & -1 & 1 & 7 & -1 \\ \hline & 1 & -1 & -7 & 1 & -6 \end{array}$$

not $x+1$.

Method 2: Use long division first.

step 1: Divide

$$\begin{array}{r|l} & x + 0 \\ \hline x^2-1 & x^3+0x^2+5x+3 \\ & x^3 \quad\quad -x \\ \hline & \quad\quad 6x+3 \end{array}$$

$$f(x) = x + \frac{6x+3}{x^2-1}$$

step 2: Differentiate using power and quotient rules.

$$f'(x) = 1 + \frac{(x^2-1)\cdot\frac{d}{dx}(6x+3) - (6x+3)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= 1 + \frac{(x^2-1)\cdot 6 - (6x+3)\cdot 2x}{[(x-1)(x+1)]^2}$$

$$= \frac{(x^2-1)^2 + 6x^2-6 - 12x^2-6x}{(x-1)^2(x+1)^2}$$

$$= \frac{x^4-2x^2+1 - 6x^2-6x-6}{(x-1)^2(x+1)^2}$$

$$\boxed{f'(x) = \frac{x^4-8x^2-6x-5}{(x-1)^2(x+1)^2}}$$

Math 250

Find derivatives

(6) a)

$$y = \frac{5x^2 - 3}{4}$$

Rewrite $y = \frac{5}{4}x^2 - \frac{3}{4}$

Differentiate using power rule, constant multiple rule, and constant rule

$$y' = \frac{5}{4} \cdot 2x^{2-1} - 0$$

$$\boxed{y' = \frac{5}{2}x}$$

* CAUTION *
If the denominator is a constant, the Quotient Rule will be painful and icky. Use monomial division to rewrite, then differentiate

b)

$$y = \frac{10}{3x^3}$$

Rewrite $y = \frac{10}{3}x^{-3}$

Differentiate using
- const multiple
- power

rules

$$y' = \frac{10}{3} \cdot (-3)x^{-3-1}$$

$$y' = -10x^{-4}$$

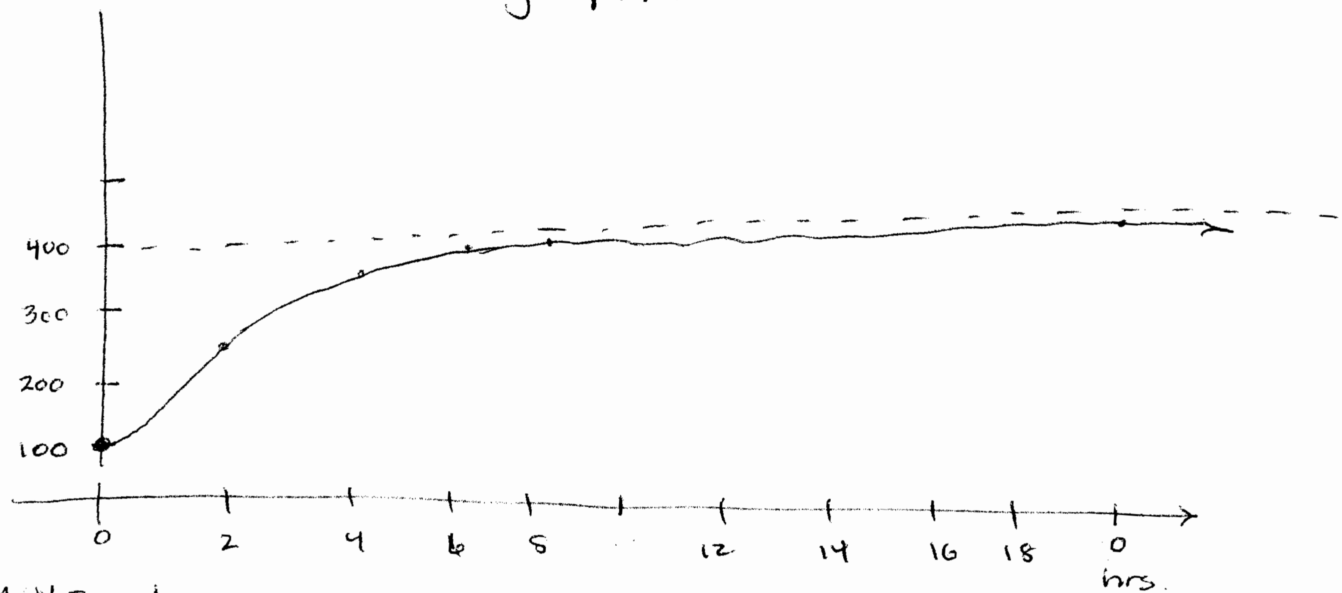
$$\boxed{y' = \frac{-10}{x^4}}$$

* CAUTION *
If the denominator can be written as a negative exponent in numerator, the quotient rule can be avoided

M250 3.4

(7) $p(t) = 400\left(\frac{t^2+1}{t^2+4}\right)$

a) Use GC to sketch graph.



XMIN = -1
XMAX = 10
XSC = 1

YMIN = -1
YMAX = 460
YSCL = 100

t	p(t)
1	160
2	250
3	308
4	340
5	359
6	370
7	377
8	382
9	386
10	388
20	397
30	399

TABLE

b) Approximate when the rate of change is greatest.

Find $p'(t)$, which gives the instantaneous rate of change at time t.

$$p'(t) = \underbrace{400}_{\text{constant multiple}}\underbrace{\left[\frac{(t^2+4)(2t)-(t^2+1)(2t)}{(t^2+4)^2}\right]}_{\text{Quotient Rule}}$$

$$= 400\left[\frac{2t^3+8t-2t^3-2t}{(t^2+4)^2}\right]$$

$$= 400\left[\frac{6t}{(t^2+4)^2}\right]$$

$$Y_2 = \frac{2400t}{(t^2+4)^2}$$

Add to graph above in GC

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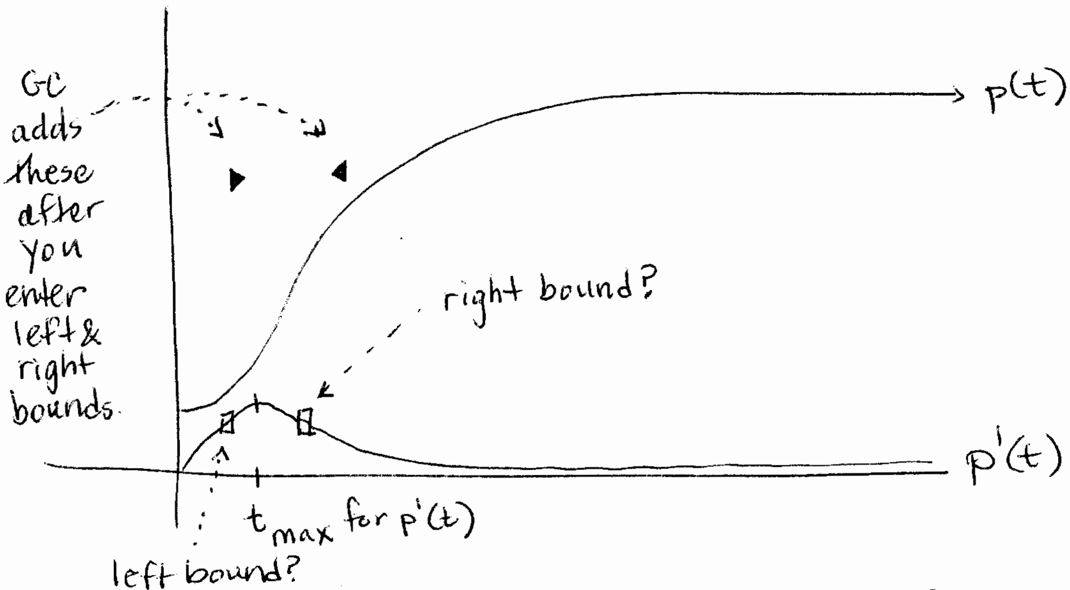
Approximate the highest value (maximum) of $p'(t)$.

$\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$

4. Maximum



Check that cursor is on the graph of $p'(t)$, not $p(t)$!
Use ▲ or ▼ to change which function is used.

Move cursor left of max, press enter to answer "Left Bound?". (Use ◄ or ►)

Move cursor right of max, press enter to answer "Right Bound?"

If the max is between your left and right bounds, the right bound will work as a guess. Or, you can move the cursor closer to the max.
Press enter to answer "Guess?"

Maximum

X = 1.1547003   Y = 97.427858

b) An approximate time of maximum growth is $\boxed{1.15 \text{ hr}}$

c) Find the steady-state population.
The steady-state population is $\lim_{t \to \infty} p(t)$.

From the graph, this appears to be 400.

Math 250 3.4

$$\lim_{t\to\infty} 400\left(\frac{t^2+1}{t^2+4}\right)$$

$$= \lim_{t\to\infty} 400 \frac{\left(\frac{t^2}{t^2}+\frac{1}{t^2}\right)}{\left(\frac{t^2}{t^2}+\frac{4}{t^2}\right)}$$

deg of denom used to find power function to be divided

$$= \lim_{t\to\infty} \frac{400\left(1+\frac{1}{t^2}\right)}{\left(1+\frac{4}{t^2}\right)}$$

core result

$$\lim_{t\to\infty} \frac{1}{t^n} = 0$$

for $n>0$

$$= 400\frac{(1+0)}{(1+0)}$$

$$= 400$$

The steady-state population is $\boxed{\text{400 cells}}$

Math 250

(8) Find derivatives

a) $f(x) = (3x^2 - 2x^{-1})\left(\dfrac{x^3 + 5x}{2}\right)$

Notice that 2 is a constant. Actually, it's $\frac{1}{2}$, and a constant multiple.

$$f(x) = \frac{1}{2}\underbrace{(3x^2 - 2x^{-1})(x^3 + 5x)}$$

If we FOIL this, we can avoid using the product rule! Hurray!

$$f(x) = \frac{1}{2}(3x^5 + 15x^3 - 2x^2 - 10)$$

Now, differentiate using only the Power Rule and the Constant Multiple Rule.

$$f'(x) = \frac{1}{2}(3 \quad + 15 \cdot 3 \cdot x^{3-} - 2 \cdot 2x^{2-1} - 0)$$

$\uparrow$ 10 is a constant

$$f'(x) = \frac{1}{2}(15x^4 + 45x^2 - 4x)$$

Distribute $= \boxed{\frac{15}{2}x^4 + \frac{45}{2}x^2 - 2x}$

or fully factor $= \boxed{\frac{x}{2}(15x^3 + 45x - 4)}$

Math 250

(8) Find derivatives — MORE PAINFUL METHODS

a)

$$f(x) = (3x^2 - 2x^{-1})\left(\frac{x^3+5x}{2}\right)$$

a) FOIL first, then differentiate
b) Use product rule, then simplify.

a)
$$f(x) = (3x^2 - 2x^{-1})\left(\tfrac{1}{2}x^3 + \tfrac{5}{2}x\right)$$
$$= \tfrac{3}{2}x^5 + \tfrac{15}{2}x^3 - x^2 - 5$$

$$f'(x) = \tfrac{15}{2}x^4 + \tfrac{45}{2}x^2 - 2x$$

Factor

$$= \boxed{\tfrac{1}{2}x\left(15x^3 + 45x - 4\right)}$$

b)
$$f'(x) = (3x^2 - 2x^{-1})\cdot\frac{d}{dx}\left(\tfrac{1}{2}x^3 + \tfrac{5}{2}x\right) + \left(\tfrac{1}{2}x^3 + \tfrac{5}{2}x\right)\cdot\frac{d}{dx}\left(3x^2 - 2x^{-1}\right)$$
$$= (3x^2 - 2x^{-1})\left(\tfrac{3}{2}x^2 + \tfrac{5}{2}\right) + \left(\tfrac{1}{2}x^3 + \tfrac{5}{2}x\right)\left(6x + 2x^{-2}\right)$$

FOIL
$$= \tfrac{9}{2}x^4 + \tfrac{15}{2}x^2 - 3x - 5x^{-1} + 3x^4 + x + 15x^2 + 5x^{-1}$$

$$= \tfrac{15}{2}x^4 + \tfrac{45}{2}x^2 - 2x$$

$$= \boxed{\tfrac{1}{2}x\left(15x^3 + 45x - 4\right)}$$

Please tell me that the product rule was painful and icky and you will always FOIL and polynomial (or polynomial-like) objects before differentiating!

M250

8 cont — Less painful method

b) Find $f'(x)$ if $f(x) = (2\sqrt{x}+1)\left(\frac{2-x}{x^2+3x}\right)$

$\sqrt{x} = x^{1/2}$

$\sqrt{x}\cdot x = x^{1/2+1} = x^{3/2}$

$$= \frac{4x^{1/2} - 2x^{3/2} + 2 - x}{x^2+3x}$$

FOIL numerator to avoid product rule

Quotient Rule

$$f'(x) = \frac{(x^2+3x)\cdot\frac{d}{dx}\left(4x^{1/2}-2x^{3/2}+2-x\right) - \left(4x^{1/2}-2x^{3/2}+2-x\right)\cdot\frac{d}{dx}(x^2+3x)}{(x^2+3x)^2}$$

$$= \frac{(x^2+3x)(2x^{-1/2}-3x^{1/2}-1) - (4x^{1/2}-2x^{3/2}+2-x)(2x+3)}{(x^2+3x)^2}$$

multiply

$$= \frac{2x^{3/2}-3x^{5/2}-x^2+6x^{1/2}-9x^{3/2}-3x-\left(8x^{3/2}+12x^{1/2}-4x^{5/2}-6x^{3/2}+4x+6-2x^2-3x\right)}{(x^2+3x)^2}$$

$$= \frac{-7x^{3/2}-3x^{5/2}-x^2+6x^{1/2}-3x-\left(2x^{3/2}+12x^{1/2}-4x^{5/2}-x+6-2x^2\right)}{(x^2+3x)^2}$$

$$= \frac{-3x^{5/2}-x^2-7x^{3/2}-3x+6x^{1/2}+4x^{5/2}+2x^2-2x^{3/2}-x-12x^{1/2}-6}{(x\cdot(x+3))^2}$$

$$= \boxed{\frac{x^{5/2}+x^2-9x^{3/2}-4x-6x^{1/2}-6}{x^2(x+3)^2}}$$

M250

Method 2: MORE PAIN ☹

(a) b) Find $f'(x)$ if $f(x) = (2\sqrt{x}+1)\left(\dfrac{2-x}{x^2+3x}\right)$

{No factors cancel. ☹}

Use product rule and quotient rule.

Step 1: Use product rule. [Hint: write structure $\frac{d}{dx}$ first.]

$$f'(x) = (2\sqrt{x}+1)\cdot \underbrace{\frac{d}{dx}\left[\frac{2-x}{x^2+3x}\right]} + \left(\frac{2-x}{x^2+3x}\right)\cdot\frac{d}{dx}(2\sqrt{x}+1)$$

Step 2: Use quotient rule. Rewrite $\sqrt{x}$ as $x^{1/2}$ for derivative

$$f'(x) = (2\sqrt{x}+1)\left[\frac{(x^2+3x)\cdot\frac{d}{dx}(2-x) - (2-x)\cdot\frac{d}{dx}[x^2+3x]}{(x^2+3x)^2}\right]$$

$$+ \left(\frac{2-x}{x^2+3x}\right)\cdot\frac{d}{dx}(2x^{1/2}+1)$$

Step 3: Find derivatives using power rule. Factor denom.

$$f'(x) = (2\sqrt{x}+1)\left[\frac{(x^2+3x)\cdot(-1) - (2-x)(2x+3)}{[x(x+3)]^2}\right] + \frac{2-x}{\cancel{x}(x+3)}\left(x^{-1/2}\right)$$

Step 4: Simplify, write with CD, factor completely.

$$f'(x) = (2\sqrt{x}+1)\left[\frac{-x^2-3x-[4x+6-2x^2-3x]}{x^2(x+3)^2}\right] + \frac{2-x}{x\sqrt{x}(x+3)}$$

$$= (2\sqrt{x}+1)\left[\frac{-x^2-3x+2x^2-x-6}{x^2(x+3)^2}\right] + \frac{2-x}{x\sqrt{x}(x+3)}$$

$$= \frac{(2\sqrt{x}+1)(x^2-4x-6)}{x^2(x+3)^2}\cdot\frac{\sqrt{x}}{\sqrt{x}} + \frac{(2-x)}{x\sqrt{x}(x+3)}\cdot\frac{x(x+3)}{x(x+3)}$$

$$= \frac{x^{1/2}(2x^{5/2}-8x^{3/2}-12x^{1/2}+x^2-4x-6) + x(2x+6-x^2-3x)}{x^2\sqrt{x}(x+3)^2}$$

$$= \frac{2x^3-8x^2-12x+x^{5/2}-4x^{3/2}-6x^{1/2}-x^3-x^2+6x}{x^2\sqrt{x}(x+3)^2}$$

M250

8 cont

$$f'(x) = \frac{x^3 - 9x^2 - 6x + x^{5/2} - 4x^{3/2} - 6x^{1/2}}{x^2 x^{1/2} (x+3)^2}$$

$$= \frac{x^{1/2}(x^{5/2} - 9x^{3/2} - 6x^{1/2} + x^2 - 4x - 6)}{x^{1/2}\ x^2 (x+3)^2}$$

$$= \boxed{\frac{x^{5/2} - 9x^{3/2} - 6x^{1/2} + x^2 - 4x - 6}{x^2 (x+3)^2}}$$

M250

(8) c) $f(x) = \dfrac{x \cos x}{x^3+2}$ ← numerator is a product! $h(x) = x$ times $g(x) = \cos x$

fraction is a quotient!

We need both the product rule and the quotient rule – but which one first?

Is the product inside the quotient or is the quotient inside the product?

The product $x \cdot \cos x$ is inside the quotient so we write the structure of the quotient rule first.

Here's where the product goes.

$$f'(x) = \frac{(\text{bottom})\cdot\left(\frac{d}{dx}\text{top}\right) - (\text{top})\left(\frac{d}{dx}\text{bottom}\right)}{(\text{bottom})^2}$$

$$= \frac{(x^3+2)\left((\text{first})\left(\frac{d}{dx}\text{second}\right)+(\text{second})\left(\frac{d}{dx}\text{first}\right)\right) - (x\cos x)(3x^2)}{(x^3+2)^2}$$

$$= \frac{(x^3+2)(x\cdot(-\sin x)+\cos x\cdot(1)) - 3x^3\cos x}{(x^3+2)^2}$$

$$= \frac{(x^3+2)(-x\sin x+\cos x) - 3x^3\cos x}{(x^3+2)^2}$$

$$= \frac{-x^4\sin x - 2x\sin x + x^3\cos x + 2\cos x - 3x^3\cos x}{(x^3+2)^2}$$

$$= \boxed{\frac{-x^4\sin x - 2x\sin x - 2x^3\cos x + 2\cos x}{(x^3+2)^2}}$$

M250 Bonus - more Painful ☹

9 $f(x) = \sqrt[3]{x}(\sqrt{x}+3)$ Find $f'(x)$ by product rule.

$f(x) = x^{1/3}(x^{1/2}+3)$

$f'(x) = \frac{d}{dx}(x^{1/3}) \cdot (x^{1/2}+3) + \frac{d}{dx}(x^{1/2}+3) \cdot x^{1/3}$

$= \frac{1}{3}x^{-2/3}(x^{1/2}+3) + \frac{1}{2}x^{-1/2} \cdot x^{1/3}$

$= \frac{1}{3}x^{-2/3}(x^{1/2}+3) + \frac{1}{2}x^{-1/6}$

$-\frac{1}{2}+\frac{1}{3}$

$\frac{-3+2}{6} = \frac{-1}{6}$

factor out least powers

$-2/3 < -1/6$!!

factor LCM of denoms $\frac{1}{3}, \frac{1}{2} \rightarrow \frac{1}{6}$

factor GCF (none) of numerators.

$= \frac{1}{6}\left(2x^{-2/3}(x^{1/2}+3) + 3x^{-1/6}\right)$ LCM denoms

$= \frac{1}{6}x^{-2/3}\left(2(x^{1/2}+3) + 3x^{-1/6-(-2/3)}\right)$

$-\frac{1}{6}+\frac{2}{3}$

$= \frac{1}{6}x^{-2/3}(2x^{1/2}+6+3x^{1/2})$

$= -\frac{1}{6}+\frac{4}{6}$

$= \frac{3}{6} = \frac{1}{2}$

$= \boxed{\frac{1}{6x^{2/3}}(5x^{1/2}+6)}$

M250

Bonus

9. $f(x) = \sqrt[3]{x}(\sqrt{x}+3)$ [We can use product rule, but why?]

step 1: Rewrite radicals using exponents, then distribute.

$f(x) = x^{1/3}(x^{1/2}+3)$

$f(x) = x^{1/3+1/2} + 3x^{1/3}$

$f(x) = x^{5/6} + 3x^{1/3}$

step 2: Differentiate using the power rule.

$f'(x) = \frac{5}{6}x^{5/6-1} + 3\cdot\frac{1}{3}\cdot x^{1/3-1}$

$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$

step 3: Factor out least powers

Note: $-2/3 < -1/6$ so $x^{-2/3}$ is the least power

$f'(x) = x^{-2/3}\left[\frac{5x^{-1/6}}{6x^{-2/3}} + \frac{x^{-2/3}}{x^{-2/3}}\right]$

$= x^{-2/3}\left[\frac{5}{6}x^{1/2} + 1\right]$

$-\frac{1}{6} - -\frac{2}{3}$

$= -\frac{1}{6} + \frac{4}{6}$

$= \frac{3}{6}$

$= \frac{1}{2}$

$\boxed{f'(x) = \frac{5\sqrt{x}+6}{6\sqrt[3]{x^2}}}$

combine with common denominator; write with radicals to match original question.